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Do Bosons Condense in a Homogeneous Magnetic Field?

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It has been known since the paper[26] and then due to a rigorous result[3] that the answer to the question in the title is negative for a three-dimensional "ideal gas of charged bosons". The present paper adds a new rigorous result in this direction. We show that the answer to the question becomes positive, if this "ideal gas of charged bosons" is simultaneously embedded in an appropriate periodic external potential. We prove that it is true for the Perfect Bose Gas (PBG), as well as for the Imperfect Bose Gas with a Mean-Field repulsive particle interaction.

KEY WORDS: Bose-Einstein Condensation; Magnetic Field; Landau Levels; Periodic External Potential; Perfect/Imperfect Bose Gas.

1. INTRODUCTION

The mathematical model of an "ideal gas of charged bosons" was invented almost fifty years ago by Schafroth[26] to study the Meissner-Ochsenfeld (M-O) effect of the magnetic field repulsion from superconductors. This abstraction not only exhibits the M-O effect but it also shows that the "ideal gas of charged bosons" in a non-zero homogeneous magnetic field do not condense unless dimensionality $d > 4$, see refs. 26 and 20.

The first *rigorous* result in this direction was due to Angelescu and Corciovei[3], who studied both Bose and Fermi perfect gases. One of their

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conclusion is a sort of "no-go" theorem forbidding the Bose-Einstein Condensation (BEC) of the "ideal gas of charged bosons" in dimension $d=3$. Since at the same time their result allows the BEC at higher dimensions ($d>4$), it is clear that the impact of the homogeneous magnetic field reduces to a *modification* of the one-particle density of states at the bottom of the spectrum due to *Landau levels*.

We are not going to discuss the physical literature related to perfect and interacting bosons in magnetic fields, we try instead to tackle this problem *rigorously* by pointing out that even simplified models could show some striking features. The aim of the present paper is to find an external ("electric") potential which is able to *restore* the BEC of the "ideal gas of charged bosons" in $d=3$. Motivated by recent experiments with optical lattices (see e.g. ref. 6 and references therein) we construct a class of *periodic external potentials* with this property.

Here it is appropriate to warn that even for a so-called "rational flux case" the *fibering* of the magnetic hamiltonian in the presence of a periodic external potential (known since ref. 8 and ref. 32) does not automatically imply that a nontrivial band structure will be formed from the Landau levels. This means that the *magnetic Bloch* "bands" introduced by[8] and[32] may degenerate into infinitely degenerate point spectrum (*constant* branches), see refs. 19 and 7 for discussion. In this paper we construct a family of external periodic potentials showing a *nondegenerate* banding of the Landau levels (*nonconstant* branches) such that the one-particle density states of the lowest band insures the BEC.

Now we come to our *mathematical model*. Denote by $\Lambda_1 \in \mathbb{R}^d$ an open, convex and simply connected domain with smooth boundary $\partial\Lambda_1$, containing the origin of coordinates; here $1 \leqslant d \leqslant 3$. The box which traps our system is given by ($L>1$)

$$\Lambda_L := \{\mathbf{x} \in \mathbb{R}^d,\ \mathbf{x}/L \in \Lambda_1\}. \tag{1.1}$$

In this paper we consider continuous $\mathbb{Z}^d$-periodic external potentials V (i.e. $\boldsymbol{\gamma} \in \mathbb{Z}^d$, $V(\mathbf{x}+\boldsymbol{\gamma}) = V(\mathbf{x})$, $\mathbf{x} \in \mathbb{R}^d$), V_L denotes the restriction of V to Λ_L. If $d=3$ we also consider a magnetic vector potential of the form:

$$\mathbf{a}(\mathbf{x}) = \omega \mathbf{a}_0(\mathbf{x}), \quad \omega \geqslant 0 \tag{1.2}$$

where either one of the two types of gauge: *symmetric* (*transverse*), $\mathbf{a}_0(\mathbf{x}) = 1/2(-x_2, x_1, 0)$ or *Landau*, $\mathbf{a}_1(\mathbf{x}) = (0, x_1, 0)$ will be used; in both cases this generates a unit magnetic field "parallel to the third direction".

Now let

$$h_L = h_L(\omega) = (-i\nabla - \mathbf{a})^2 + V_L, \tag{1.3}$$

be the one particle Hamiltonian defined on $L^2(\Lambda_L)$ with *Dirichlet boundary conditions* (DBC) on $\partial\Lambda_L$. Then h_L has *purely discrete* spectrum[24], we denote the set of eigenvalues (counting multiplicities and in increasing order) by $\{\lambda_j\}_{j\geqslant 1}$ and by $\{u_j\}_{j\geqslant 1}$ the corresponding set of eigenfunctions. We denote by h_∞ the unique self-adjoint extension of the operator $(-i\nabla - \mathbf{a})^2 + V$ defined on $C_0^\infty(\mathbb{R}^d)$[23]. Because of the magnetic field, the nature of the spectrum of h_∞ is not known in general; but since by our assumptions h_∞ is bounded from below and commutes with the (magnetic) translations, then h_∞ has *no discrete spectrum*. Let us denote by $E_0 := \inf \sigma(h_\infty)$. Moreover, due to standard arguments involving the min-max principle, for all $L > 1$ we have $E_0 \leqslant \lambda_1$[24].

We first consider a *perfect Bose gas* (PBG) confined in the volume Λ_L, each particle of the gas interacts with the background potential V_L and the external magnetic field. The case of an *imperfect* gas with a *Mean-Field* (MF) type of particle potential will be considered in Section 5. In the grand-canonical ensemble, the pressure of a perfect gas at inverse temperature $\beta > 0$ and chemical potential $\mu < E_0$ is given by the well known expression, see e.g.[16] and also Section 5:

$$p_L(\beta, z) := -\frac{1}{\beta|\Lambda_L|}\sum_{j\geqslant 1}\ln(1 - ze^{-\beta\lambda_j}), \tag{1.4}$$

where $\{\lambda_j\}_{j\geqslant 1}$ is the set of eigenvalues of the one particle Hamiltonian (1.3); z is the fugacity $z := e^{\beta\mu}$. The density of the gas is:

$$\rho_L(\beta, z) := \beta z \frac{\partial p_L}{\partial z}(\beta, z) = \frac{1}{|\Lambda_L|}\sum_{j\geqslant 1}\frac{ze^{-\beta\lambda_j}}{1 - ze^{-\beta\lambda_j}}. \tag{1.5}$$

Since the semigroup $e^{-\beta h_L}$ generated by h_L is trace class, i.e. $\sum_{j\geqslant 1} e^{-\beta\lambda_j} < \infty$[27], the series in (1.4) and (1.5) are absolutely convergent. It is known that under our assumptions the thermodynamic limit ($L \to \infty$) of the pressure p_L and of the particle density ρ_L exist[3] and we are now interested in the behavior of $\rho_\infty(\beta, z) := \lim_{L\to\infty} \rho_L(\beta, z)$ near the critical value $z_c = e^{\beta E_0}$, $\beta > 0$ since this determines whether the Bose-Einstein condensation takes place for our system[16,33].

Let $P_I(h_L)$ be the spectral projection of the operator h_L for a Borel set $I \subset \mathbb{R}$. Denoting by $N_L(\lambda) := \mathrm{Tr}\,(P_{(-\infty,\lambda)}(h_L))$ the counting function of eigenstates of h_L (the number of eigenstates of h_L for eigenvalues less than λ), we have:

$$\rho_L(\beta, z) = -\int_{E_0}^{\infty} \left[\partial_\lambda \frac{z e^{\beta\lambda}}{1 - z e^{\beta\lambda}} \right] \frac{N_L(\lambda)}{|\Lambda_L|} d\lambda. \tag{1.6}$$

Recall that the integrated density of states for h_∞, denoted by $n_\infty(\lambda)$ is defined as a weak limit:

$$n_\infty(\lambda) = \lim_{L\to\infty} \frac{N_L(\lambda)}{|\Lambda_L|} \tag{1.7}$$

on the space of continuous functions $C_0([E_0, \infty))$, see e.g.[22].

Moreover, let χ_{Λ_L} be the characteristic function of Λ_L and $P_I(h_\infty)$ be the spectral projection of h_∞ for a Borel set $I \subset \mathbb{R}$. Then under even more general conditions than ours, for any $\lambda \in \mathbb{R} \setminus \sigma_p(h_\infty)$, the pointwise limit

$$\tilde{n}_\infty(\lambda) := \lim_{L\to\infty} \frac{\mathrm{Tr}(\chi_{\Lambda_L} P_{(-\infty,\lambda)}(h_\infty) \chi_{\Lambda_L})}{|\Lambda_L|} \tag{1.8}$$

exists, is continuous and coincides with $n_\infty(\lambda)$ (see e.g. refs. 5, 15 and 13).

Notice that by (1.6) and (1.7), the density $\rho_L(\beta, z)$ admits for $z < z_c$ a thermodynamic limit of the form:

$$\rho_\infty(\beta, z) = -\int_{E_0}^{\infty} \left[\partial_\lambda \frac{z\, e^{-\beta\lambda}}{1 - z\, e^{-\beta\lambda}} \right] n_\infty(\lambda) d\lambda. \tag{1.9}$$

We easily see from (1.9) that the limit density $\rho_\infty(\beta, z = e^{\beta\mu})$, increases with μ and decreases with β. Moreover, $\rho_\infty(\beta, \cdot)$ has an analytic extension to the domain $\mathbb{C} \setminus [z_c, \infty)$.

Definition 1.1. A homogeneous Bose gas manifests the Bose-Einstein condensation (BEC) if for every $\beta > 0$, it admits a finite critical density $\rho_c(\beta)$, where

$$\rho_c(\beta) := \lim_{\mu \nearrow E_0} \rho_\infty(\beta, z = e^{\beta\mu}) < \infty. \tag{1.10}$$

Correspondingly, the critical temperature $1/\beta_c(\rho)$ for a given density ρ is defined as the unique solution of the equation $\rho = \rho_c(\beta)$, i.e.

$$\rho = \rho_c(\beta_c(\rho)).$$

For the "free" PBG, when $\omega = 0$ and $V = 0$, the integrated density of states is known explicitly $n_\infty(\lambda) = [(2\sqrt{\pi})^d \Gamma(1 + d/2)]^{-1} \lambda^{d/2}$, see e.g. ref. 24. Hence, by (1.9) one gets $\rho_c(\beta) < \infty$ for $d > 2$. This implies the BEC of the perfect gas for these dimensions.

On the other hand, we know from[3] that for $d = 3$ PBG, the BEC does not exist (i.e. $\rho_c(\beta) = \infty$) in the presence of a homogeneous magnetic field ($\omega \neq 0$, $V = 0$). We shall see that this is related to the fact that

$$n_\infty(\lambda) = B_{\omega,d} \cdot (\lambda - E_0(\omega))^{d/2-1} + o((\lambda - E_0(\omega))^{d/2-1}) \tag{1.11}$$

for $\lambda \searrow E_0(\omega)$. Hence, integral (1.9) diverges for $z = z_c$, if $d = 3$.

In what follows, we will show that by adding a certain external periodic potential, we can restore the BEC in our system with magnetic field. In particular, we prove the following main theorem:

Theorem 1.2. Consider a three-dimensional PBG in a homogeneous magnetic field, where the one-particle Hamiltonian is given by $h_{0,L} = (-i\nabla - \mathbf{a})^2$ in $L^2(\Lambda_L)$ with DBC on $\partial\Lambda_L$; here $\mathbf{a} = \omega \mathbf{a}_0$ where $\mathbf{a}_0(\mathbf{x}) := 1/2(-x_2, x_1, 0)$, and $\omega > 0$. Assume that V is a $\mathbb{Z}^3-$periodic and continuous external potential, and define the one-particle Hamiltonian $h_L = h_{0,L} + V_L$ in $L^2(\Lambda_L)$.

(i) Assume that $\omega > 0$ is arbitrary, and V is independent of either one of the "transverse" variables x_1 or x_2. Then the BEC is absent (see for details Proposition 3.1 and Remark 3.3).

(ii) Assume that $\omega = 2\pi$, and V depends non-trivially on both x_1 and x_2. Then there exists a fairly large class of such potentials, for which the perturbed system described by h_L manifests the BEC (see for details (3.15) and Remark 3.6).

(iii) The previous result remains true by switching on a mean-field particle self-interaction. For a precise statement, see Theorem 4.3.

The proof of Theorem 1.2 is based on the following remark. Since we have $h_L \geqslant h_{0,L} + \min(V)$, the min-max principle implies that the j-th eigenvalue of h_L is larger or equal than the j-th eigenvalue of $h_{0,L}$ shifted with $\min(V)$. Then with obvious notation we have:

$$N_L(\lambda) \leqslant N_{0,L}(\lambda - \min(V)),$$

thus by using (1.7) and (1.11) we get a polynomial upper bound $\sim \lambda^{d/2-1}$ for $n_\infty(\lambda)$ at infinity. Therefore, the only factor which can decide whether the limit in (1.10) is finite or not is the behavior of $n_\infty(\lambda)$ near the bottom E_0 of the spectrum $\sigma(h_\infty)$. Indeed, one can easily see that a sufficient condition for having a finite critical density is the estimate:

$$n_\infty(\lambda) \leqslant const \cdot (\lambda - E_0)^{1+\alpha}, \quad \lambda \in (E_0, E_0 + \epsilon) \tag{1.12}$$

for some $\alpha > 0$ and finite $\epsilon > 0$. On the contrary, a sufficient condition for an infinite critical density (or zero critical temperature) is the estimate

$$n_\infty(\lambda) \geqslant const \cdot (\lambda - E_0), \quad \lambda \in (E_0, E_0 + \epsilon) \tag{1.13}$$

for some finite $\epsilon > 0$.

Remark 1.3. More generally, a necessary and sufficient condition for having a finite PBG critical density for every $\beta > 0$ and $d = 3$ is the following estimate:

$$\int_{E_0}^{E_0+1} \frac{n_\infty(\lambda)}{(\lambda - E_0)^2} d\lambda < \infty. \tag{1.14}$$

This condition also implies that $\rho_c : (0, \infty) \to (0, \infty)$ is smooth and invertible (one shows that ρ_c is decreasing and onto).

By the virtue of Definition 1.1, an equivalent way of defining BEC is imposing that the critical temperature $\beta_c(\rho)^{-1}$ exists and is positive for every density. If the critical density is infinite, we set $\beta_c(\rho) = \infty$.

Remark 1.4. To clarify why the existence problem for BEC in magnetic field is so nontrivial, it is not out of place to mention here the following points:

(a) Let $\omega = 0$ and let the PBG be placed in a continuous $\mathbb{Z}^d$-periodic potential V. Then one can check (1.14) for the lowest band of absolutely continuous spectrum corresponding to the lowest nonconstant branch in the fiber decomposition of the one-particle periodic Schrödinger operator, see Propositions 2.1 and 2.2 in Section 2. We stress that this result is true for all periodic and continuous V's.

(b) If $\omega > 0$, then (1.14) is not satisfied when $V = 0^{(3)}$. Moreover, in this paper we show that there exists a large class of $\mathbb{Z}^d$-periodic potentials $V \neq 0$ which cannot make the integral in (1.14) bounded, see Theorem 1.2 (i) and Section 3.1. But it is bounded, when $\omega = 0$, by (a).

(c) If we assume that $\omega = 2\pi N$, for some integer N, then the magnetic translations form an abelian group and this allows again to write the magnetic Hamiltonian with $\mathbb{Z}^d$-periodic potential V as a direct fiber integral[8,32]. Similar to (a), the fiber operator has only discrete eigenvalues depending on the fiber parameter (quasi momentum),i.e. the branches. An immediate consequence is that the spectrum of the magnetic Hamiltonian will consist either from infinitely degenerate eigenvalues (constant branches), or absolutely continuous bands (nonconstant branches); the singular continuous spectrum is excluded. So, it sounds as one has more control of the spectrum comparing to (b). But in spite of a general belief that for rational flux, a nontrivial magnetic Bloch band structure will be formed from Landau levels, it is still an unsolved mathematical problem to give sufficient conditions which guarantee the absence of infinitely degenerate point spectrum for the magnetic Hamiltonian in this case. In other words, to prove that eigenvalues of the fiber Hamiltonian are nonconstant branches as functions of quasi momentum, i.e. that the former Landau levels indeed produce absolutely continuous "mini-bands" as soon as one switches on some (well-tuned) periodic potential, see Section 3.2. This is again in contrast to the nonmagnetic case (a), for which this problem was settled by Thomas, see ref. 24 and Section 2, or to the zero-flux case which was only recently solved in[29] by a rather sophisticated method.

(d) In the present paper (Section 3.2 and Appendix 2) we construct for $\omega = 2\pi$ a family of external periodic potentials producing in the fiber decomposition a number of nonconstant branches. More important, the integrated density of states near the lowest band will obey (1.14).

We conclude this section by giving some technical points which are important in this paper. Let $f \in C_0^\infty(\mathbb{R})$, we have, (see ref. 13):

$$\lim_{L\to\infty} \frac{1}{|\Lambda_L|} \mathrm{Tr}\left[\chi_{\Lambda_L} f(h_\infty)\chi_{\Lambda_L}\right] = -\int_{\mathbb{R}} f'(t) n_\infty(t)dt. \tag{1.15}$$

Moreover, we will show in Appendix 1 that $f(h_\infty)$ is an integral operator with a smooth integral kernel $f_{h_\infty}(\mathbf{x}, \mathbf{x}')$. Since h_∞ commutes with the magnetic translations (see Appendix 2, (5.36) for their definition), then:

$$\forall \boldsymbol{\gamma} \in \mathbb{Z}^d, \quad f_{h_\infty}(\mathbf{x}+\boldsymbol{\gamma}, \mathbf{x}+\boldsymbol{\gamma}) = f_{h_\infty}(\mathbf{x}, \mathbf{x}), \quad \forall \mathbf{x} \in \mathbb{R}^d. \tag{1.16}$$

Therefore

$$\begin{aligned}\lim_{L\to\infty} \frac{1}{|\Lambda_L|} \mathrm{Tr}\left[\chi_{\Lambda_L} f(h_\infty)\chi_{\Lambda_L}\right] &= -\int_{\mathbb{R}} f'(t) n_\infty(t)dt \\ &= \frac{1}{|\Omega|}\int_\Omega f_{h_\infty}(\mathbf{x}, \mathbf{x})d\mathbf{x},\end{aligned} \tag{1.17}$$

where $\Omega := (-1/2, 1/2)^d$ is the elementary cell, see Section 2.

Assume that the operator h_∞ is (magnetic) translation invariant in some subspace $\mathbb{R}^{d'}$ of $\mathbb{R}^d, d'<d$. For all $\mathbf{x}\in\mathbb{R}^d$ we write $\mathbf{x}=(\overline{x},x)$, where $\overline{x}$ is the component of $\mathbf{x}$ in the subspace $\mathbb{R}^{d'}$. The kernel's diagonal of f_{h_∞} then is $\overline{x}$ independent. Thus (1.17) reads as

$$\lim_{L\to\infty}\frac{1}{|\Lambda_L|}\mathrm{Tr}\left[\chi_{\Lambda_L}f(h_\infty)\chi_{\Lambda_L}\right]=\frac{1}{|\Xi|}\int_\Xi f_{h_\infty}((\overline{0},x);(\overline{0},x))dx \qquad (1.18)$$

where now $\Xi\subset\mathbb{R}^{d-d'}$ is the elementary cell in the subspace orthogonal to $\mathbb{R}^{d'}$.

Our paper is organized as follows: Section 2 is devoted to a discussion on various results concerning the BEC for a *perfect* Bose gas in the presence of *periodic* external potential *without* magnetic field. Most of the facts given in this section are known but they are instructive for the rest of the paper. In Section 3 we discuss the stability of the BEC after an external magnetic field is switched on; there we will also prove the *first part* of Theorem 1.2. The results of Section 3 are applied in Section 4 where we study the *imperfect Bose gas*, in the case of a *mean-field type* interaction. For reader's convenience we collect in the two Appendices of Section 5 some technical results indispensable for the proofs placed in Section 3.

2. BEC FOR A BOSE GAS IN PERIODIC EXTERNAL POTENTIALS

In this section we assume that each particle of the Bose gas interacts with a continuous, $\mathbb{Z}^d$-periodic potential V; without loss of generality we will choose $\min(V)=0$. The one particle Hamiltonian $h_L=-\Delta+V$ is then a self-adjoint operator on $L^2(\Lambda_L)$ (with DBC on $\partial\Lambda_L$) as well as the infinite-volume Hamiltonian $h_\infty=-\Delta+V$ on $L^2(\mathbb{R}^d)$.

We now apply the standard Floquet theory for periodic operators (see ref. 24). Let $\Omega^*=2\pi\Omega=(-\pi,\pi)^d\subset\mathbb{R}^d$ be the elementary cell of the lattice dual to $\mathbb{Z}^d$, which is generated by translations of the cell $\Omega=(-1/2,1/2)^d$. Define a unitary operator:

$$\begin{aligned} &U:L^2(\mathbb{R}^d)\mapsto\int_{\Omega^*}^{\oplus}L^2(\Omega)d\mathbf{k},\\ &(Uf)(\mathbf{k},\underline{x}):=\sum_{\boldsymbol{\gamma}\in\mathbb{Z}^d}\frac{1}{(2\pi)^{d/2}}e^{-i\mathbf{k}\cdot(\underline{x}+\boldsymbol{\gamma})}f(\underline{x}+\boldsymbol{\gamma}),\end{aligned}$$

where $\mathbf{k}\in\Omega^*$ and $\underline{x}\in\Omega$. Then the unitary transformation of h_∞ is decomposable into the direct integral: $Uh_\infty U^*=\int_{\Omega^*}^{\oplus}h(\mathbf{k})d\mathbf{k}$. Here the fiber Hamiltonians:

$$h(\mathbf{k}) = (-i\nabla + \mathbf{k})^2 + V, \quad \mathbf{k} \in \Omega^* \tag{2.1}$$

live in $L^2(\Omega)$ with periodic boundary conditions. They have purely discrete spectrum which accumulates at infinity; for a given $\mathbf{k} \in \Omega^*$, we denote the set of corresponding eigenvalues by $\{\lambda_j(\mathbf{k})\}_{j \geqslant 1}$ and branches by $\{\lambda_j(\mathbf{k})\}_{j \geqslant 1, \mathbf{k} \in \Omega^*}$.

An important ingredient for us is a result due to Kirsch and Simon[18] about the band (branch) structure of $\sigma(h_\infty)$:

Proposition 2.1. Assume that the potential V is $\mathbb{Z}^d$-periodic, continuous and $\min(V) = 0$. Consider the operator $h_\infty = -\Delta + V$ on $L^2(\mathbb{R}^d)$ and let $\{\lambda_j(\mathbf{k})\}_{j \geqslant 1}$ be the eigenvalues of the fiber Hamiltonians $h(\mathbf{k}), \mathbf{k} \in \Omega^*$, defined in (2.1). Denoting by $E_0 = \inf \sigma(H_\infty) \geqslant 0$, we have:

(i) The lowest branch $\lambda_1(\mathbf{k})$ has E_0 as a nondegenerate minimum at $\mathbf{k} = \mathbf{0}$ i.e.

$$\min_{\mathbf{k} \in \Omega^*} \lambda_1(\mathbf{k}) = \lambda_1(\mathbf{0}) = E_0, \quad \lambda_1(\mathbf{k}) = E_0 + Q(\mathbf{k}, \mathbf{k}) + o(|\mathbf{k}|^2), \quad \text{for } \mathbf{k} \to \mathbf{0}, \tag{2.2}$$

where Q is a positive quadratic form on $\mathbb{R}^d$;

(ii) E_0 is isolated from the rest of the spectrum $\sigma(h_\infty)$ i.e.

$$\inf_{j \geqslant 2} \{\min_{\mathbf{k} \in \Omega^*} \lambda_j(\mathbf{k})\} - E_0 = : \lambda_0 > 0. \tag{2.3}$$

The eigenvalues of the quadratic form Q are related to the so-called *effective* masses in the corresponding directions.

The main result of this section is contained in the following statement:

Proposition 2.2. Under the same assumptions as in Proposition 2.1, the critical density defined as the limit in (1.10) is infinite for $d = 1$ or $d = 2$. If $d = 3$, the critical density (1.10) is finite and the PBG manifests the BEC.

Proof. We show that if $d \in \{1, 2\}$ then (1.13) holds while if $d = 3$ then (1.12) holds. Let $f \in C_0^\infty(\mathbb{R})$. Since the kernel $f_{h_\infty}(\underline{x}, \underline{x}')$ is jointly continuous and decay polynomially with respect to the variable $\underline{x} - \underline{x}'$ (see Appendix 1), the operator $f(h(\mathbf{k}))$ admits an integral kernel which due to the fiber decomposition has for $\underline{x}$ and $\underline{x}' \in \Omega$ the representation:

$$f_{h(\mathbf{k})}(\underline{x}, \underline{x}') = \sum_{\gamma \in \mathbb{Z}^d} f_{h_\infty}(\underline{x} + \gamma, \underline{x}')\, e^{-i\mathbf{k} \cdot (\underline{x} - \underline{x}' + \gamma)}. \tag{2.4}$$

We will see in the next section an extension of this formula for the more general magnetic case (cf. (3.17)). Now (2.4) implies that $f_{h(\mathbf{k})}(\underline{x}, \underline{x}')$ is jointly continuous in $\underline{x}$ and $\underline{x}'$. On the other hand $f(h(\mathbf{k}))$ is a finite rank operator and due to the smoothness property evoked above, its trace equals the integral of its kernel's diagonal:

$$\begin{aligned} \operatorname{Tr} f(h(\mathbf{k})) &= \sum_{j \geqslant 1} f(\lambda_j(\mathbf{k})) = \int_\Omega f_{h(\mathbf{k})}(\underline{x}, \underline{x}) d\underline{x} \\ &= \sum_{\boldsymbol{\gamma} \in \mathbb{Z}^d} \int_\Omega f_{h_\infty}(\underline{x} + \boldsymbol{\gamma}, \underline{x}) \, e^{-i\mathbf{k}\cdot\boldsymbol{\gamma}} d\underline{x}. \end{aligned} \tag{2.5}$$

Then integrating (2.5) with respect to the $\mathbf{k}$ variable we have:

$$\sum_{j \geqslant 1} \frac{1}{(2\pi)^d} \int_{\Omega^*} f(\lambda_j(\mathbf{k})) d\mathbf{k} = \int_\Omega f_{h_\infty}(\underline{x}, \underline{x}) d\underline{x} = -\int_{\mathbb{R}} f'(t) n_\infty(t) dt, \tag{2.6}$$

where the second equality comes from (1.17). Take a weakly converging sequence $f_n(t) \to \chi_{[E_0-0,\lambda]}(t)$, $n \to \infty$, and $\lambda \leqslant \max_{\mathbf{k} \in \Omega^*} \lambda_1(\mathbf{k})$. Then (2.6) and Proposition 2.1 imply:

$$n_\infty(\lambda) = \frac{1}{(2\pi)^d} \int_{\Omega^*} \chi_{[E_0-0,\lambda]}(\lambda_1(\mathbf{k})) d\mathbf{k}, \tag{2.7}$$

where χ_I denotes the indicator of the set $I \subset \mathbb{R}$. Now, Proposition 2.1 (i) and a change of variables in (2.7) give:

$$n_\infty(\lambda) = A_d (\lambda - E_0)^{d/2} + o((\lambda - E_0)^{d/2}), \tag{2.8}$$

for $\lambda \searrow E_0$, where $A_d > 0$ by virtue of (2.2), i.e. by positivity of the curvature of the branch $\lambda_1(\mathbf{k})$ at $\mathbf{k} = \mathbf{0}$. Notice that this is exactly the same behaviour as in the "free" case (i.e. $V = 0$), and the proposition is proven. ∎

3. BEC FOR A BOSE GAS IN PRESENCE OF A CONSTANT MAGNETIC FIELD

In the next subsection we prove the *first part* of our main Theorem 1.2. As it has been mentioned before, we are motivated by the work of Angelescu-Corcovei[3] who showed that for a free, three-dimensional Bose gas, the critical density is infinite in the presence of a constant magnetic field, i.e. BEC disappears. The mechanism of that is described in Section 1: by creating the Landau levels, the magnetic fields leads to increasing of the integrated density of states at the bottom of the spectrum $\sigma(h_0(\omega \neq 0))$. We only consider the case of dimension $d=3$, for which the BEC in the PBG holds when $\mathbf{a}=0$ even for a periodic external potential by Proposition 2.2. We first give an example of a periodic potential V where the BEC is *destroyed* by *any* constant magnetic field. Then in the last subsection we show that this is not always the case, i.e. we prove the *second part* of Theorem 1.2.

This means that in contract to the free Bose-gas there is no compulsory elimination of the BEC by the constant magnetic field in the presence of a properly tuned periodic potential.

3.1. Instability of BEC in the Presence of a Magnetic Field

We start with a *simple case* where the continuous external potential $V(\mathbf{x}) = v(x_1)$ i.e. it depends only on the *one* variable and v is $\mathbb{Z}$-periodic. Throughout this section, we use the Landau gauge $\mathbf{a}_1(\mathbf{x}) = (0, x_1, 0) \in \mathbb{R}^3$. Notice that the choice of a particular gauge is irrelevant since the density of states is gauge invariant. Under these conditions, the "bulk" Hamiltonian is:

$$h_\infty = (-i\nabla - \omega \mathbf{a}_1)^2 + v = -\partial_{x_1}^2 + v(x_1) + (-i\partial_{x_2} - \omega x_1)^2 - \partial_{x_3}^2, \quad (3.1)$$

acting on $L^2(\mathbb{R}^3)$, where $\omega \geqslant 0$.

Proposition 3.1. Consider a perfect Bose gas described by the one particle Hamiltonian h_L defined as the restriction of the operator (3.1) to $L^2(\Lambda_L)$ with DBC. Then for every $\omega > 0$, the critical density is infinite, i.e. the BEC is destroyed.

Before coming to the proof of Proposition 3.1, we need some technical results. We will often write a vector $\mathbf{u} \in \mathbb{R}^3$ as $\mathbf{u} = (u_1, \tilde{u})$ with $\tilde{u} = (u_2, u_3) \in \mathbb{R}^2$. Decompose $L^2(\mathbb{R}^3)$ with the help of the partial Fourier transform with respect to x_2 and x_3:

$$U : L^2(\mathbb{R}^3) \mapsto \int_{\mathbb{R}^2}^{\oplus} L^2(\mathbb{R}) d\tilde{k}, \; U = \int_{\mathbb{R}^2}^{\oplus} U_{\tilde{k}} d\tilde{k},$$
$$(U_{\tilde{k}} f)(t) = \frac{1}{2\pi} \int_{\mathbb{R}^2} e^{-i\tilde{k}\tilde{x}} f(t, \tilde{x}) d\tilde{x}. \quad (3.2)$$

Then

$$U h_\infty U^* = \int_{\mathbb{R}^2}^{\oplus} h(\tilde{k}) d\tilde{k}, \quad h(\tilde{k}) = -\frac{d^2}{dt^2} + (\omega t - k_2)^2 + v(t) + k_3^2. \quad (3.3)$$

If $z \in \mathbb{C} \setminus \mathbb{R}$, denote by $(h_\infty - z)^{-1}(x_1, \tilde{x}; x_1', \tilde{x}')$ and $[h(\tilde{k}) - z]^{-1}(t, t')$ the integral kernels of the corresponding operators. We are interested here in the analog of (2.4).

Lemma 3.2. If $f \in C_0^\infty(\mathbb{R})$, then $f(h_\infty)$ admits a smooth integral kernel $f_{h_\infty}(t, \tilde{y}; t', \tilde{y}')$ and we have the representation

$$f_{h(\tilde{k})}(t, t') = \int_{\mathbb{R}^2} e^{-i\tilde{k}\tilde{y}} f_{h_\infty}(t, \tilde{y}; t', \mathbf{0}) d\tilde{y}, \quad (3.4)$$

for the kernel of the operator in (3.3) where the integral is absolutely convergent.

Proof. Let $f \in C_0^\infty(\mathbb{R})$. Then $f(h_\infty)$ admits a smooth integral kernel which decays faster than any polynomial in $\bar{y}$ for t and t' fixed[14,27] (see also Appendix 1). Moreover, by standard arguments[27], it is enough to prove (3.4) for the resolvent operator. Let $g \in C_0^\infty(\mathbb{R}^3)$. Then we have:

$$\begin{aligned}[U_{\tilde{k}}(h_\infty - z)^{-1} U^* g](t) \;=\; & \int_{\mathbb{R}} dt' \int_{\mathbb{R}^2} d\tilde{x}' \int_{\mathbb{R}^2} d\tilde{x} \frac{e^{-i\tilde{k}\tilde{x}}}{2\pi} (h_\infty - z)^{-1}(t, \tilde{x}; t', \tilde{x}') \\ & \times \int_{\mathbb{R}^2} d\tilde{k}' \frac{e^{i\tilde{k}'\tilde{x}'}}{2\pi} g(t', \tilde{k}'). \end{aligned} \quad (3.5)$$

The above integral makes sense because $(h_\infty - z)^{-1}(t, \tilde{x}; t', \tilde{x}')$ decays exponentially in $|\tilde{x} - \tilde{x}'|$ for t and t' fixed (see ref. 14). Since h_∞ commutes with translations in both directions x_2 and x_3, we get:

$$(h_\infty - z)^{-1}(t, \tilde{x}; t', \tilde{x}') = (h_\infty - z)^{-1}(t, \tilde{x} - \tilde{x}'; t', \mathbf{0}).$$

Then the integrals in (3.5) take the form

$$[U_{\tilde{k}}(h_\infty - z)^{-1} U^* g](t) = \int_{\mathbb{R}} dt' \left\{ \int_{\mathbb{R}^2} e^{-i\tilde{k}\tilde{y}} (h_\infty - z)^{-1}(t, \tilde{y}; t', \mathbf{0}) d\tilde{y} \right\} g(t', \tilde{k}). \quad (3.6)$$

By virtue of (3.2) and (3.3), this yields the equality

$$[h(\tilde{k})-z]^{-1}(t,t')=\int_{\mathbb{R}^2} e^{-i\tilde{k}\tilde{y}}(h_\infty - z)^{-1}(t,\tilde{y};t',\mathbf{0})d\tilde{y}, \tag{3.7}$$

which has to be understood as equality between smooth functions outside the diagonal $t=t'$. ∎

Proof of Proposition 3.1. By the conditions on the external potential $v(t)$, the fiber operator $h(\tilde{k})$ has purely discrete spectrum for any $\tilde{k}\in\mathbb{R}^2$.

We denote by $\{\lambda_n(k_2)\}_{n\geqslant 1}$ the nondegenerate eigenvalues of the operator $h(k_2,0)=-d^2/dt^2+(\omega t-k_2)^2+v(t)$, and by $\{\psi_n(\cdot,k_2)\}_{n\geqslant 1}$ the corresponding eigenfunctions. Let $f\in C_0^\infty(\mathbb{R})$. Then we have

$$f_{h(\tilde{k})}(t,t')=\sum_{n\geqslant 1} f(\lambda_n(k_2)+k_3^2)\psi_n(t,k_2)\overline{\psi}_n(t',k_2). \tag{3.8}$$

Here the sum over n is finite (f has compact support), but $\lim_{n\to\infty}\lambda_n(k_2)=\infty$ uniformly in $k_2\in\mathbb{R}$. Hence, $f_{h(\tilde{k})}$ is a finite-rank operator. This can be explicitly seen from the fact that the fiber operator $h(\tilde{k})$ in (3.3) is unitarily equivalent to the operator:

$$-\frac{d^2}{dt^2}+\omega^2t^2+v(t+k_2/\omega)+k_3^2,$$

which is a harmonic oscillator plus a bounded perturbation. Moreover, this representation makes evident that $\mathbb{Z}$-periodicity of v implies $\mathbb{Z}_\omega$-periodicity of $\lambda_n(k_2)$ for all $n\geqslant 1$.

Notice that we are only interested in what happens near the bottom of the spectrum, $E_0=\inf\sigma(h_\infty)=\inf_{k_2}\lambda_1(k_2)$. Because of the non-degeneracy of the eigenvalues $\{\lambda_n(k_2)\}_{n\geqslant 1}$, E_0 is isolated from the other bands, i.e from $\mathrm{Ran}(\lambda_n)$ with $n\geqslant 2$. Applying usual arguments (see ref. 24), ψ_1 can be chosen positive. Hence, if f is supported close enough to E_0, by (3.8) we obtain:

$$f_{h(\tilde{k})}(t,t)=f(\lambda_1(k_2)+k_3^2)\,\psi_1^2(t,k_2).$$

Then taking the Fourier transform in (3.4) we get

$$\begin{aligned}\frac{1}{4\pi^2}\int_{\mathbb{R}^2} f_{h(\tilde{k})}(t,t)d\tilde{k} &= \frac{1}{4\pi^2}\int_{\mathbb{R}^2} f(\lambda_1(k_2)+k_3^2)\,\psi_1^2(t,k_2)d\tilde{k}\\ &= f_{h_\infty}(t,\mathbf{0};t,\mathbf{0}).\end{aligned} \tag{3.9}$$

Note that the second integral in (3.9) converges, since by standard methods we can prove that ψ_1 is sharply localized near k_2/ω and that it has a gaussian decay of the form $const\, e^{-\alpha(t-k_2/\omega)^2}$ (see ref. 2). Now using (1.17), (1.18) for λ close to E_0, we eventually get (in a way similar to (2.7)) that (3.9) implies

$$n_\infty(\lambda) = \frac{1}{4\pi^2}\int_\Xi\int_{\mathbb{R}^2}\chi_{[E_0-0,\lambda]}(\lambda_1(k_2)+k_3^2)\,\psi_1^2(t,k_2)dt\,d\tilde{k}, \tag{3.10}$$

where $\Xi=[-1/2,1/2]$.

In the particular case when $v=0$, we have $E_0=\omega=\lambda_1(k_2)$. Fix λ between the first two Landau levels: $\lambda\in(\omega,3\omega)$. Then by integrating with respect to k_2 we obtain

$$\int_{\mathbb{R}}\psi_1^2(t,k_2)dk_2=\omega,$$

thus $n_\infty(\lambda)=\omega(\lambda-\omega)^{1/2}/2\pi^2$ for the pure magnetic case. Notice the it is the "bad" exponent 1/2 that makes the critical density to diverge, see (1.13).

Now we show that even if $v\neq 0$, the integrated density of states still behaves like in (1.13). First, the general theory insures that the branch $\lambda_1(k_2)$ is a real analytic function of k_2[24]. If it is a *constant*, then we are essentially back to the case $v=0$, since one has a lower bound for n_∞ of the form:

$$\begin{aligned} n_\infty(\lambda) \geqslant{}& \frac{1}{4\pi^2}\left(\int_{\mathbb{R}}\chi_{[E_0-0,\lambda]}(E_0+k_3^2)dk_3\right)\\ &\times\inf_{-1/2\leqslant t\leqslant 1/2}\int_{\mathbb{R}}\psi_1^2(t,k_2)dk_2, \end{aligned}\tag{3.11}$$

for every $\lambda\in(E_0,E_0+\epsilon)$. Since $\psi_1^2(t,k_2)$ is jointly smooth in both arguments and positive, the mapping

$$[-1/2,1/2]\ni t\mapsto\int_{\mathbb{R}}\psi_1^2(t,k_2)dk_2\in\mathbb{R}$$

has a positive minimum, so $n_\infty(\lambda)\geqslant a(\lambda-E_0)^{1/2}+o((\lambda-E_0)^{1/2})$ for $\lambda\searrow E_0$ and for some $a>0$, i.e. we get back to case (1.13).

Let the branch $\lambda_1(k_2)$ be not a *constant*. Since it is a real analytic function and $\mathbb{Z}_\omega$-periodic, there exists a finite set of points $\{\xi_1,\dots,\xi_N\}\subset[0,\omega)$, where λ_1 takes its minimal value E_0. Let $\lambda_1(\xi_j)=E_0$, $j\in\{1,\dots,N\}$.

Then there exists a positive integer $n_j \geqslant 1$ and a constant C_j so that for k_2 close to ξ_j

$$\lambda_1(k_1) \sim E_0 + C_j(k_2 - \xi_j)^{2n_j}.$$

To get a lower bound for $n_\infty(\lambda)$ we may take the integral (3.10) with respect to $\tilde{k}$ over compact domains around the minima of the function $\lambda_1(k_2)$. In fact, for λ close to E_0 we can bound n_∞ from below by taking into account just one of those minima:

$$\begin{aligned} n_\infty(\lambda) \geqslant\ & const\left(\int_{-1/2}^{1/2} \psi_1^2(t, \xi_j)dt\right) \\ & \cdot \int_{\mathbb{R}^2} \chi_{[E_0,\lambda]}(E_0 + \delta C_j(k_2 - \xi_j)^{2n_j} + k_3^2)\, d\tilde{k}, \end{aligned} \tag{3.12}$$

for some $\delta > 1$. This leads to

$$n_\infty(\lambda) \geqslant const \cdot (\lambda - E_0)^{\frac{1}{2} + \frac{1}{2n_j}},$$

which clearly implies (1.13) for $\lambda - E_0$ small enough. Therefore, the proposition is proven. ∎

Remark 3.3. Proposition 3.1 can be easily extended to $\mathbb{Z}^2$-periodic potentials $v = v(x_1, x_3)$. In this case, a similar analysis shows that the corresponding Hamiltonian h_∞ is unitarily equivalent to the operator $\int_{\mathbb{R}\times(-\pi,\pi)} h(\tilde{k})d\tilde{k}$, where now

$$h(\tilde{k}) = -\partial_t^2 + (\omega t - k_2)^2 + (-i\partial_s + k_3)^2 + v(t, s) \tag{3.13}$$

on $L^2(\mathbb{R} \times (-1/2, 1/2))$ with periodic boundary conditions on $\mathbb{R} \times \{\pm 1/2\}$. Notice that for every $\tilde{k} \in \mathbb{R} \times (-\pi, \pi)$, the fiber Hamiltonian $h(\tilde{k})$ has a compact resolvent which is positivity improving (see ref. 24). Let $\{\lambda(\tilde{k})\}_{n\geqslant 1}$ denote the set of eigenvalues of $h(\tilde{k})$ and $\{\psi_n(\cdot, \tilde{k})\}_{n\geqslant 1}$ be the corresponding eigenvectors. Then $\lambda_1(\tilde{k})$ is continuous and nondegenerate for any $\tilde{k}$. Let E_0 be the minimal value of $\lambda_1(\tilde{k})$. Then there exists a point $(\xi, \zeta) \in \mathbb{R} \times (-\pi, \pi)$, $(n, m) \in \mathbb{N}^2$ and $(C, D) \in \mathbb{R}^2$ two non-negative constants such that in the neighborhood of (ξ, ζ) we have the expansion

$$\begin{aligned} \lambda_1(\tilde{k}) =\ & E_0 + C(k_2 - \xi)^{2n} + D(k_3 - \zeta)^{2m} \\ & + o((k_2 - \xi)^{2n} + (k_3 - \zeta)^{2m}). \end{aligned} \tag{3.14}$$

Notice that by a standard Thomas' argument (see ref. 24) concerning the k_3 variable one concludes that $\lambda_1(\tilde{k})$ cannot be constant in k_3, which implies $D>0$ and $m \geqslant 1$.

If we choose λ close to E_0, formula (3.10) now takes the form

$$n_\infty(\lambda) = \frac{1}{2\pi}\int_{\Xi^2} ds dt \int_{\mathbb{R}\times(-\pi,\pi)} d\tilde{k}\, \chi_{[E_0-0,\lambda]}(\lambda_1(\tilde{k}))\, \psi_1^2(s,t,\tilde{k}).$$

Then the rest of the reasoning follows the same lines as above. For $C=0$ in (3.14), we use the argument as the one for (3.11), while for $C>0$ we take the estimate (3.12). This gives

$$n_\infty(\lambda) \geqslant const\,(\lambda - E_0)^{\frac{1}{2n}+\frac{1}{2m}},$$

for $\lambda \searrow E_0$, which implies (1.13), even for non-degenerate minimum $n=m=1$ in (3.14).

3.2. An Example of Finite Critical Density for Non-Zero Uniform Magnetic Field

The previous subsection showed that the Bose condensate can be destroyed by turning on a no matter how weak constant magnetic field. Here we want to show that this is not always true. Let dimension $d=3$. We choose in this subsection the gauge $\mathbf{a}_0(x_1,x_2) = 1/2(-x_2,x_1,0)$ and we construct an external periodic potential, which depends on all three variables such that the critical density becomes finite.

We assume that the external potential has the following form:

$$V_\epsilon(\mathbf{x}) = \epsilon\cdot[v_1(x_1)+v_2(x_2)] + v_3(x_3), \tag{3.15}$$

where $\epsilon>0$ and small, each of the functions $\{v_j\}_{j=1}^3$ is a smooth $\mathbb{Z}$-periodic potential, and we also suppose that neither one of v_1 and v_2 is constant.

Take the magnetic field intensity $\omega = 2\pi$, then the "bulk" Hamiltonian can be written as

$$h_\infty = (-i\nabla_{\mathbf{x}} - 2\pi\mathbf{a}_0(x_1,x_2))^2 + V_\epsilon = h_\epsilon \otimes \mathbf{1} + \mathbf{1}\otimes h_3, \tag{3.16}$$

where the operator $h_\epsilon = (-i\nabla - 2\pi\mathbf{a}_0)^2 + \epsilon(v_1+v_2)$ lives in $L^2(\mathbb{R}^2)$ while the operator $h_3 = -d^2/dx_3^2 + v_3$ lives in $L^2(\mathbb{R})$.

First, let us introduce some notation. We write an arbitrary vector $\mathbf{x} \in \mathbb{R}^3$ as $\mathbf{x} = (\bar{x}, x_3)$ where $\bar{x} := (x_1, x_2)$. We often use the notation $\Xi = (-1/2, 1/2)$. The elementary cell is $\Omega = \Xi^3$ and the one of the dual lattice Ω^* is given by

$$\Omega^* := \{2\pi \boldsymbol{\xi}, \quad \boldsymbol{\xi} \in \Xi^3\} = 2\pi\, \Xi^3.$$

According to Appendix 2, the operator h_∞ is unitarily equivalent to $\int_{\Xi^3} h(\boldsymbol{\xi}) d\boldsymbol{\xi}$, where the fiber operator can be further written as

$$h(\boldsymbol{\xi}) = h_\epsilon(\bar{\xi}) \otimes \mathbf{1} + \mathbf{1} \otimes h_3(\xi_3).$$

Here the operators $h_\epsilon(\bar{\xi}) = h_0(\bar{\xi}) + \epsilon V(\bar{x})$, $h_0(\bar{\xi}) := [-i\nabla_{\bar{x}} - \mathbf{a}(\bar{x}) + \mathbf{k}(\bar{\xi})]^2$ live in $L^2(\Xi^2)$ with *"magnetic" periodic boundary conditions* (see Appendix 2 for definition), and $h_3(\xi_3) = (-id/dx_3 + \xi_3)^2 + v_3(x_3)$ in $L^2(\Xi)$.

Recall that $\mathbf{a}(\bar{x}) = 2\pi \mathbf{a}_0(\bar{x})$ and $\mathbf{k}(\bar{\xi}) = 2\pi(\mathbf{e}_1\xi_1 + \mathbf{e}_2\xi_2)$. If f is a $C_0^\infty(\mathbb{R})$-function then the integral kernel of $f_{h(\boldsymbol{\xi})}$ is given by

$$\begin{aligned} f_{h(\boldsymbol{\xi})}(\bar{x}, x_3; \bar{x}', x_3') = &\sum_{\bar{\gamma} \in \mathbb{Z}^2} \sum_{\gamma_3 \in \mathbb{Z}} e^{-i\phi(\bar{x}, \bar{\gamma}) - ib(\bar{\gamma}) - 2\pi i \bar{\xi}\cdot(\bar{x} + \bar{\gamma} - \bar{x}')} e^{-2\pi i \xi_3 (x_3 + \gamma_3 - x_3')} \\ &\times f_{h_\infty}(\bar{x} + \bar{\gamma}, x_3 + \gamma_3; \bar{x}', x_3'), \end{aligned} \tag{3.17}$$

for every $\mathbf{x}, \mathbf{x}' \in \Omega$, where $\phi(\bar{x}, \bar{\gamma}) = \pi(x_2 n - x_1 m)$ and $b(\bar{\gamma}) = \pi mn$ for $\bar{\gamma} = m\mathbf{e}_1 + n\mathbf{e}_2$, see Appendix 2. Notice that the third coordinate is not influenced by the magnetic field, while the first two coordinates are essentially treated in the Appendix 2, see for instance (5.47). Then by integrating the trace of $f_{h(\boldsymbol{\xi})}$ with respect to $\boldsymbol{\xi}$ we obtain

$$\int_{\Xi^3} \operatorname{Tr} f_{h(\boldsymbol{\xi})} d\boldsymbol{\xi} = \int_{\Xi^3} \int_\Omega f_{h(\boldsymbol{\xi})}(\mathbf{x}, \mathbf{x}) d\mathbf{x} d\boldsymbol{\xi} = \int_\Omega f_{h_\infty}(\mathbf{x}, \mathbf{x}) d\mathbf{x}. \tag{3.18}$$

Now, since we put the magnetic flux through Ξ^2 to be exactly 2π, all eigenvalues of $h_\epsilon(\bar{\xi})$ are simple and belong to an interval of width of order ϵ around the former ($\epsilon = 0$) *Landau levels*. These are well-known *magnetic Bloch* "bands"[(8)] and[(32)]. As we indicated in Introduction, in contrast to nonmagnetic case, there are no clear conditions available under which those eigenvalues (branches) are not *constants*, thus generating an infinitely degenerate point spectrum. Transformation of this spectrum into *absolutely continuous* bands (*banding* of Landau levels) produces nontrivial magnetic Bloch bands, see Introduction and Appendix 2.

Denote them by (see also (5.54))

$$\{2\pi(2n+1)+\epsilon a_n(\epsilon,\overline{\xi}),\ \ \overline{\xi}\in\Xi^2\}_{n\geqslant 0} \tag{3.19}$$

and by $l_m(\xi_3)$, $m\geqslant 1$ the eigenvalues of $h_3(\xi_3)$. Then the spectrum of h_∞ is given by the closure of the range of the function:

$$2\pi(2n+1)+\epsilon a_n(\epsilon,\overline{\xi})+l_m(\xi_3),\quad \boldsymbol{\xi}=(\overline{\xi},\xi_3)\in\Xi^3,\quad n\geqslant 0, m\geqslant 1.$$

Notice that by Proposition 2.1 we know that the branch l_1 reaches its minimum at zero and $l_1(\xi_3)\sim l_1(0)+C\xi_3^2$ in its neighborhood. Then the bottom of the spectrum $\sigma(h_\infty)$ is equal to

$$E_0=2\pi+\epsilon\min_{\overline{\xi}\in\Xi^2}a_0(\epsilon,\overline{\xi})+l_1(0).$$

Moreover, similar to the nonmagnetic case E_0 is isolated from the other bands if ϵ is *small enough*. Thus, we can repeat our arguments leading to the estimate of the integral in (2.7) for $\lambda\searrow E_0$. This gives for the integrated density of states in this limit the exponent 3/2, provided the minimum of $a_0(\epsilon,\cdot)$ is nondegenerate. (See Lemma 3.4 for the definition of a nondegenerate minimum for $\epsilon=0$).

This would prove the *second* part of Theorem 1.2. Thus we continue by the following statement.

Lemma 3.4. Assume $a_0(0,\cdot)$ has a nondegenerate (local) minimum at $\overline{\xi}_0\in\Xi^2$, i.e. there exists a symmetric and positive matrix $Q\in\mathcal{M}_2(\mathbb{R})$ such that for small $|\overline{\xi}-\overline{\xi}_0|$ we have

$$a_0(0,\overline{\xi})=a_0(0,\overline{\xi}_0)+\langle\overline{\xi}-\overline{\xi}_0,Q(\overline{\xi}-\overline{\xi}_0)\rangle+\mathcal{O}(|\overline{\xi}-\overline{\xi}_0|^3). \tag{3.20}$$

Here $\langle\cdot,\cdot\rangle$ is the scalar product in $\mathbb{C}^2$.

Then for ϵ small enough, there exists $\overline{\xi}_\epsilon$ close to $\overline{\xi}_0$ such that the functional $a_0(\epsilon,\cdot)$ has $\overline{\xi}_\epsilon$ as nondegenerate minimum.

Proof. By assumption, we have $\overline{\xi}_0\in\Xi^2$ so that $(\nabla_{\overline{\xi}}a_0)(0,\overline{\xi}_0)=\mathbf{0}$ while the differential at $\overline{\xi}_0$ given by $[D_{\overline{\xi}}(\nabla_{\overline{\xi}}a_0)](0,\overline{\xi}_0)=Q$ is invertible. Due to the smoothness properties of $a_0(\epsilon,\overline{\xi})$ with respect to all its variables, one can easily verify the hypotheses of the *Implicit Function Theorem*, which finishes the proof. ∎

Remark 3.5. Assume that $a_0(0,\cdot)$ has a finite number of nondegenerate local minima, only. Then due to the joint analyticity of a_0 with respect to $(\epsilon, \overline{\xi})$, there exists $\epsilon_0 > 0$ small enough such that for $|\epsilon| < \epsilon_0$, the absolute minimum of $a_0(\epsilon,\cdot)$ will also be nondegenerate and there are no other critical points than the ones given by Lemma 3.4.

Therefore, the only thing which remains to be studied is the behaviour of $a_0(0,\cdot)$ near its absolute minimum. But (see Appendix 2), we have already got a fairly explicit expression for it in (5.55), where we have now to put $V(x_1, x_1) = v_1(x_1) + v_2(x_1)$, $\omega = 2\pi$ and $n = 0$. We use the notation

$$(\hat{v})_\gamma = \int_{-1/2}^{1/2} e^{-2\pi i \gamma x} v(x)dx, \quad \gamma \in \mathbb{Z},$$

for the discrete Fourier transform, to obtain, by virtue of (5.55), that

$$\begin{aligned} a_0(0,\overline{\xi}) &= b_{0,1}(\overline{\xi}) + b_{0,2}(\overline{\xi}), \\ b_{0,1}(\overline{\xi}) = b_{0,1}(\xi_2) &= \sum_{\gamma_2 \in \mathbb{Z}} e^{-2\pi i \xi_2 \gamma_2} e^{-\pi \gamma_2^2/2} (\hat{v}_1)_{\gamma_2}, \\ b_{0,2}(\overline{\xi}) = b_{0,2}(\xi_1) &= \sum_{\gamma_1 \in \mathbb{Z}} e^{2\pi i \xi_1 \gamma_1} e^{-\pi \gamma_1^2/2} (\hat{v}_2)_{\gamma_1}. \end{aligned} \tag{3.21}$$

It is easy to see that by a judicious choice of v_1 and v_2 we can create any profile we want for the functions $b_{0,1}$ and $b_{0,2}$. In particular, we can make the local minima nondegenerate. Indeed, choose two nonconstant functions $p, q : (-1/2, 1/2) \mapsto \mathbb{R}$ which admit C^∞-extensions to $\mathbb{R}$; assume they have nondegenerate absolute minima correspondingly at $\xi_{0,p}$ and $\xi_{0,q}$ in the interval $(-1/2, 1/2)$. Denote by

$$\tilde{p}_s(x) = \sum_{k=-s}^{s} e^{2\pi i x k} (\hat{p})_k, \quad s > 1, x \in (-1/2, 1/2)$$

the approximation of p by its first $2s+1$ Fourier components. Since p is smooth, then for $s = M$ large enough the approximation $\tilde{p}_M$ will have a nondegenerate absolute minimum at $\xi_{M,p}$ close to $\xi_{0,p}$. Define

$$v_1(x) := \sum_{k=-M}^{M} e^{2\pi i x k} e^{\pi k^2/2} (\hat{p})_k, \; x \in (-1/2, 1/2).$$

Then by (3.21) it follows that $b_{0,1}(\xi_2) = \tilde{p}_M(-\xi_2)$. Thus, $b_{0,1}$ has a nondegenerate absolute minimum at $-\xi_{M,p}$. Similar line of reasoning involving the function q implies the same conclusion for $b_{0,2}$. This finishes the proof of the *second* part of Theorem 1.2.

Remark 3.6. By inspection of this line of reasoning, one finds that there is no need to restrict the potential periodic in directions (x_1, x_2) to a separable form, see (3.15). In fact our proof goes through verbatim for $\mathbb{Z}^3$-periodic potential

$$\epsilon \cdot [v_1(x_1) + v_2(x_2) + \delta \cdot v(x_1, x_2)] + v_3(x_3)$$

for ϵ and δ small enough.

4. IMPERFECT BOSE GAS: MEAN-FIELD INTERACTION

To discuss whether the BEC found in the two previous sections survives the switching on of a particle interaction, we consider here the simplest version of it, known as the *Mean-Field* (MF) interaction, see e.g. ref. 31.

To this end we need the *second quantized* form of the interacting gas Hamiltonian in the boson Fock space $\mathcal{F}^B(L^2(\Lambda_L))$:

$$H_{\Lambda_L}(\mu) := H_{\Lambda_L} - \mu N_{\Lambda_L} = T_{\Lambda_L} - \mu N_{\Lambda_L} + U_{\Lambda_L}. \tag{4.1}$$

Here

$$T_{\Lambda_L} := \int_{\Lambda_L} \mathrm{d}\mathbf{x} a^*(\mathbf{x}) h_L a(\mathbf{x}) \tag{4.2}$$

is the kinetic-energy part with one-particle operator h_L defined by (1.3) and

$$U_{\Lambda_L} := \frac{1}{2} \int_{(\Lambda_L)^2} \mathrm{d}\mathbf{x}\mathrm{d}\mathbf{y}\ a^*(\mathbf{x}) a^*(\mathbf{y}) v(\mathbf{x}-\mathbf{y}) a(\mathbf{y}) a(\mathbf{x}), \tag{4.3}$$

is the interaction defined by a two-body potential $v(\mathbf{x}-\mathbf{y})$, where $a^*(\mathbf{x})$, $a(\mathbf{x})$ are the usual *boson-field* operators,

$$N_{\Lambda_L} := \int_{\Lambda_L} \mathrm{d}\mathbf{x}\, a^*(\mathbf{x}) a(\mathbf{x}) \tag{4.4}$$

is the particle-number operator, and μ is the chemical potential.

To ensure the existence of the thermodynamics of the Bose gas (4.1) for all parameters (β, μ) of the grand-canonical ensemble, it used to suppose that the interaction $v(\mathbf{x}-\mathbf{y})$ is superstable[25]. For example, let the pair interaction potential $v(\mathbf{x}) = v(-\mathbf{x})$ be a real, non-negative continuous function from $L^1(\mathbb{R}^d)$. Since $v \in L^1(\mathbb{R}^d)$, the Fourier transform $\hat{v}(\mathbf{q})$ exists, and

$$\hat{v}(\mathbf{0}) = \int_{\mathbb{R}^d} \mathrm{d}\mathbf{x}\ v(\mathbf{x}) > 0 \quad \text{with } \hat{v}(\mathbf{0}) \geqslant \hat{v}(\mathbf{q}),\ \ \mathbf{q} \in \mathbb{R}^d. \tag{4.5}$$

It is known [ref. 25] that the corresponding interaction is superstable, i.e. the n-body potential satisfies the inequality

$$\sum_{1 \leqslant i < j \leqslant n} v(\mathbf{x}_i - \mathbf{x}_j) \geqslant \frac{A}{2|\Lambda_L|} n^2 - Bn \tag{4.6}$$

for some constants $A > 0$, $B \geqslant 0$, for all $n \in \mathbb{N}$, $\mathbf{x}_i, \mathbf{x}_j \in \Lambda_L$ and L large enough which implies that the thermodynamic potentials exist for all values of the chemical potential μ.

To introduce the MF interaction consider the scaled potential :

$$v_\lambda(x) := \lambda^d v(\lambda x),\ \lambda \geqslant 0. \tag{4.7}$$

Denote by

$$p_{\Lambda_L}\left[H^\lambda_{\Lambda_L}\right](\beta, \mu) := \frac{1}{\beta|\Lambda_L|} \ln Tr_{\mathcal{F}^B(L^2(\Lambda_L))} e^{-\beta(H^\lambda_{\Lambda_L} - \mu N_{\Lambda_L})} \tag{4.8}$$

the grand-canonical pressure defined by the Hamiltonian $H^\lambda_{\Lambda_L}$ with the two-body interaction $v_\lambda(x)$. Then the limit

$$\lim_{\lambda\to 0} \lim_{L\to\infty} p_{\Lambda_L}\left[H^\lambda_{\Lambda_L}\right](\beta, \mu) = p^{vdW}(\beta, \mu), \tag{4.9}$$

exists and it is known as the *van der Waals limit*[11]. If one chooses the scaled two-body potential (4.7) in the form

$$v_L(\mathbf{x}) := g|\Lambda_L|^{-1}, \tag{4.10}$$

Then the limit

$$\lim_{L\to\infty} p_{\Lambda_L}\left[H^{\lambda_L}_{\Lambda_L}\right](\beta, \mu) = p^{MF}(\beta, \mu), \tag{4.11}$$

exists and it is known as the *Mean-Field limit*[11]. Notice that by virtue of (4.4) the interaction (4.3) in this case takes the form:

$$\begin{aligned} U_{\Lambda_L}^{MF} &= \frac{1}{2}\int_{(\Lambda_L)^2} \mathbf{dxdy}\; a^*(\mathbf{x})a^*(\mathbf{y})v_L(\mathbf{x}-\mathbf{y})a(\mathbf{y})a(\mathbf{x}) \\ &= \frac{1}{2}\frac{g}{|\Lambda_L|}N_{\Lambda_L}(N_{\Lambda_L}-I), \end{aligned} \tag{4.12}$$

i.e., the corresponding Hamiltonian in the Fock space $\mathcal{F}^B(L^2(\Lambda_L))$ is defined by $H_{\Lambda_L}^{MF} := T_{\Lambda_L} + U_{\Lambda_L}^{MF}$.

Since the spectrum of the one-particle kinetic-energy operator is such that $\inf\sigma(h_\infty) = E_0$, see Section 1, thermodynamic behaviour of the boson gas (4.1) is E_0-dependent.

Lemma 4.1 [Thermodynamic Functions]. The grand-canonical pressure $p^{MF,E_0}(\beta,\mu)$ (4.11) of the M-F boson gas (4.1) exists for all $\beta \geqslant 0, \mu \in \mathbb{R}$ and is given by the Legendre transformation:

$$p^{MF,E_0}(\beta,\mu) = \sup_{\rho\geqslant 0}\left(\mu\rho - f^{MF,E_0}(\beta,\rho)\right), \tag{4.13}$$

where the canonical free-energy density $f^{MF,E_0}(\beta,\rho)$ at inverse temperature β and density ρ is given by

$$f^{MF,E_0}(\beta,\rho) = f^{PBG,E_0}(\beta,\rho) + \frac{g}{2}\rho^2. \tag{4.14}$$

Here $f^{PBG,E_0}(\beta,\rho)$ is the free-energy density of the PBG, corresponding to (4.2).

Proof. The grand-canonical thermodynamic pressure of the PBG (4.2) is given by the limit

$$\begin{aligned} p^{PBG,E_0}(\beta,\mu) &= \lim_{L\to\infty} p_{\Lambda_L}^{PBG,E_0}(\beta,\mu) \\ &= \lim_{L\to\infty}\frac{1}{\beta|\Lambda_L|} Tr_{\mathcal{F}^B(L^2(\Lambda_L))} e^{-\beta(T_{\Lambda_L}-\mu N_{\Lambda_L})}, \end{aligned} \tag{4.15}$$

which implies that in order to be well defined, the chemical potential μ must be bounded from above: $\mu < E_0$, see Section 1. On the other hand, one has

$$\begin{aligned} p_{\Lambda_L}^{MF,E_0}(\beta,\mu) = \frac{1}{\beta|\Lambda_L|}\ln\sum_{N=0}^{\infty} & e^{\beta(\mu N - gN(N-1)/2|\Lambda_L|)} \\ & \times Tr_{\mathcal{F}_N^B(L^2(\Lambda_L))} e^{-\beta T_{\Lambda_L}^{(N)}}, \end{aligned} \tag{4.16}$$

where $T_{\Lambda_L}^{(N)}$ is a restriction of the kinetic-energy operator (4.3) on the N-particle sector $\mathcal{F}_N^B(L^2(\Lambda_L))$ of the Fock space $\mathcal{F}^B(L^2(\Lambda_L))$, $N=0,1,2,\ldots$. Put

$$
\begin{aligned}
f^{PBG,E_0=0}(\beta,\rho) &:= f^{PBG}(\beta,\rho),\\
p^{PBG,E_0=0}(\beta,\mu) &:= p^{PBG}(\beta,\mu).
\end{aligned}
\tag{4.17}
$$

Since the canonical free-energy density $f^{PBG,E_0}(\beta,\rho)$, is the Legendre transformation of $p^{PBG,E_0}(\beta,\mu)$, by definitions (4.17) one gets:

$$
\begin{aligned}
f^{PBG,E_0}(\beta,\rho) &= \sup_{\mu\leqslant E_0}\left(\rho\mu - p^{PBG,E_0}(\beta,\mu)\right)\\
&= \sup_{\mu\leqslant E_0}\left(\rho(\mu-E_0) - p^{PBG}(\beta,\mu-E_0) + E_0\rho\right)\\
&= f^{PBG}(\beta,\rho) + E_0\rho.
\end{aligned}
\tag{4.18}
$$

The free-energy density of the mean-field model (4.12) at temperature β and density $\rho = N/|\Lambda_L|$ is defined by

$$
f_{\Lambda_L}[H_{\Lambda_L}^{MF}](\beta,\rho) = -\frac{1}{\beta|\Lambda_L|}\ln Tr_{\mathcal{F}_N^B(L^2(\Lambda_L))} e^{-\beta H_{\Lambda_L}^{MF(N)}},
\tag{4.19}
$$

where $Tr_{\mathcal{F}_N^B(L^2(\Lambda_L))}(\cdot)$ denotes the trace over the Hilbert space $\mathcal{F}_N^B(L^2(\Lambda_L))$ of symmetrized functions for $N=\rho|\Lambda_L|$ bosons. Since $\mathcal{F}_N^B(L^2(\Lambda_L))$ is the proper space of the particle-number operator N_{Λ_L} with the proper value N, the mean-field interaction term on this space is constant. Thus, we immediately find (4.14) in the thermodynamic limit:

$$
\lim_{L\to\infty} f_{\Lambda_L}[H_{\Lambda_L}^{MF}](\beta,\rho) = \lim_{L\to\infty} f_{\Lambda_L}[T_{\Lambda_L}](\beta,\rho) + \frac{g}{2}\rho^2.
\tag{4.20}
$$

By (4.16) the pressure of the mean-field gas is well-defined for all $\mu\in\mathbb{R}$, and it is again the Legendre transform of $f^{MF}(\beta,\rho)$, yielding formula (4.13). ∎

Corollary 4.2 [Pressure of the M-F Bose Gas]. The grand-canonical pressure of a mean-field Bose Gas (4.13) is given by

$$
p^{MF,E_0}(\beta,\mu) = \begin{cases} \mu\overline{\rho}(\beta,\mu) - f^{MF,E_0}(\beta,\overline{\rho}(\beta,\mu)), & \text{for } \mu\leqslant E_0 + g\rho_c(\beta);\\ (\mu-E_0)^2/2g + p^{PBG}(\beta,0), & \text{for } \mu > E_0 + g\rho_c(\beta), \end{cases}
\tag{4.21}
$$

where $\overline{\rho}(\beta,\mu))$ is a unique solution of the chemical potential equation

$$\mu = \partial_\rho f^{MF,E_0}(\beta,\rho) = \partial_\rho f^{PBG}(\beta,\rho) + E_0 + g\rho. \tag{4.22}$$

Here $\rho^{PBG,E_0}(\beta,\mu) = \rho^{PBG}(\beta,\mu - E_0)$ is the total density of the Perfect Bose gas and $\rho^{PBG,E_0}(\beta,E_0) \equiv \rho_c(\beta)$, defined by (1.10).

Theorem 4.3. For the mean-field Bose gas (4.12), one gets the following expressions for particle densities in the thermodynamic limit. The total grand-canonical density is given by

$$\rho^{MF,E_0}(\beta,\mu) = \begin{cases} \overline{\rho}(\beta,\mu)), & \text{for} \quad \mu \leqslant E_0 + g\rho_c(\beta), \\ (\mu - E_0)/g, & \text{for} \quad \mu > E_0 + g\rho_c(\beta). \end{cases} \tag{4.23}$$

The condensate density is given by

$$\rho_0^{MF,E_0}(\beta,\mu) = \begin{cases} 0, & \text{for} \quad \mu \leqslant E_0 + g\rho_c(\beta), \\ (\mu - E_0)/g - \rho_c(\beta), & \text{for} \quad \mu > E_0 + g\rho_c(\beta). \end{cases} \tag{4.24}$$

Proof. Since the total grand-canonical density of the mean-field Bose gas is defined by thermodynamic relation $\rho^{MF,E_0}(\beta,\mu) = \partial_\mu p^{MF,E_0}(\beta,\mu)$, the part (4.23) of our theorem follows directly from (4.21) and (4.22).

The part (4.24) is a more delicate matter. It is based on the strong equivalence of ensembles for the mean-field Bose gas and the fact that expectations in the canonical ensemble coincide with those for the PBG. This implies[21,30] that the particle density in the states with the energies higher than some $\delta > 0$ is equal to

$$\begin{aligned} &\lim_{L\to\infty} \frac{1}{|\Lambda_L|} \sum_{\{j:\lambda_j > E_0+\delta\}} \left\langle a^*(u_j)a(u_j)\right\rangle_{\Lambda_L}(\beta,\mu) \\ &\quad = \int_{E_0+\delta}^{\infty} \frac{dn_\infty(\lambda)}{e^{\beta(\lambda-\mu+g\rho^{MF,E_0}(\beta,\mu))}-1}, \end{aligned} \tag{4.25}$$

where $a^*(u_j) = \int_{\Lambda_L} d\mathbf{x} u_j(\mathbf{x}) a^*(\mathbf{x}) = \left(a(u_j)\right)^*$ for the eigenvectors $\left\{u_j(x)\right\}_{j\geqslant 1}$ of the operator h_L (1.3). By virtue of (4.23) we get from (4.25) that

$$\lim_{\delta\to 0}\lim_{L\to\infty} \frac{1}{|\Lambda_L|} \sum_{\{j:\lambda_j > E_0+\delta\}} \left\langle a^*(u_j)a(u_j)\right\rangle_{\Lambda_L}(\beta,\mu) = \rho_c(\beta) < \infty \tag{4.26}$$

for $\mu > E_0 + g\rho_c(\beta)$. Since the total particle density

$$\rho^{MF,E_0}(\beta,\mu) = \lim_{L\to\infty} \frac{1}{|\Lambda_L|} \sum_{j\geqslant 0} \left\langle a^*(u_j)a(u_j)\right\rangle_{\Lambda_L}(\beta,\mu) \tag{4.27}$$

is equal to (4.23), the limit (4.26) proves the Bose-Einstein condensation (4.24) in the following form:

$$\rho_0^{MF,E_0}(\beta,\mu) = \lim_{\delta\to 0}\lim_{L\to\infty} \frac{1}{|\Lambda_L|} \times \sum_{\{j:\lambda_j\leqslant E_0+\delta\}} \left\langle a^*(u_j)a(u_j)\right\rangle_{\Lambda_L}(\beta,\mu). \tag{4.28}$$

∎

5. APPENDICES

5.1. Appendix 1: Smoothness and Decay of the Integral Kernel for f_{h_∞}

For simplicity we consider here only dimension $d=3$, and the operator $h_\infty = (-i\nabla - \mathbf{a})^2 + V$ as in Introduction. Let $f \in C_0^\infty(\mathbb{R})$. We write

$$f(h_\infty) = \exp(-h_\infty)\cdot \tilde{f}(h_\infty)\cdot \exp(-h_\infty), \quad \tilde{f}(t) = e^{2t} f(t). \tag{5.29}$$

It is well-known (see e.g. the arguments via Feynman-Kac-Itô formula in ref. 28) that $\exp(-h_\infty)$ admits an integral kernel, denoted by $e^{-h_\infty}(\mathbf{x},\mathbf{x}')$, which obeys the so-called *diamagnetic inequality*:

$$\left|e^{-h_\infty}(\mathbf{x},\mathbf{x}')\right| \leqslant e^{-\min(V)}\cdot \frac{1}{(4\pi)^{3/2}} e^{-|\mathbf{x}-\mathbf{x}'|^2/4}.$$

Moreover, let $K \in \mathbb{R}^3$ be a compact set, and $\alpha_1, \alpha_2 \in \mathbb{N}^3$. Under the smoothness conditions we assumed for $\mathbf{a}$ and V, the semigroup kernel obeys (see ref. 28)

$$\left|\partial_{\mathbf{x}}^{\alpha_1}\partial_{\mathbf{x}'}^{\alpha_2} e^{-h_\infty}(\mathbf{x},\mathbf{x}')\right| \leqslant C_1\cdot e^{-C_2|\mathbf{x}-\mathbf{x}'|} < \infty, \quad \mathbf{x}'\in\mathbb{R}^3,\ \mathbf{x}\in K, \tag{5.30}$$

where C_1 and C_2 are constants which may depend on α's and K.

By virtue of the previous estimate, we can then write down the kernel of $f(h_\infty)$ as

$$f_{h_\infty}(\mathbf{x},\mathbf{x}')=\int_{\mathbb{R}^3} e^{-h_\infty}(\mathbf{x},\mathbf{y})\left[\tilde f(h_\infty)e^{-h_\infty}(\cdot,\mathbf{x}')\right](\mathbf{y})d\mathbf{y}. \tag{5.31}$$

By the Cauchy-Schwarz inequality with respect to the $\mathbf{y}$ variable in (5.31), we get that the above kernel belongs to $L^\infty(\mathbb{R}^3\times\mathbb{R}^3)$. Since this is also true for $\tilde f$, we can then rewrite (5.31) as

$$f_{h_\infty}(\mathbf{x},\mathbf{x}')=\int_{\mathbb{R}^6} e^{-h_\infty}(\mathbf{x},\mathbf{y})\cdot \tilde f_{h_\infty}(\mathbf{y},\mathbf{y}')\cdot e^{-h_\infty}(\mathbf{y}',\mathbf{x}')d\mathbf{y}d\mathbf{y}'.$$

This together with (5.30) allow us to conclude that $f_{h_\infty}(\cdot,\cdot)\in C^\infty(\mathbb{R}^6)$.

Finally, regarding the decay of the above kernel, we recall a result of[14], which adapted to our setting assures that for every $N\geqslant 1$, there exists a positive constant $C_{N,f}$ so that

$$|f_{h_\infty}(\mathbf{x},\mathbf{x}')|\leqslant C_{N,f}\cdot(1+|\mathbf{x}-\mathbf{x}'|)^{-N},\quad \mathbf{x},\mathbf{x}'\in\mathbb{R}^3.$$

5.2. Appendix 2: Proof of 3.17 and 3.19

Many aspects treated in this Appendix are well-known, and go back to[8] and[32]. As we mentioned above, in the "rational flux case", the magnetic translations form an *abelian* group and this allows one to write the magnetic Hamiltonian as a *direct fiber integral*. The fiber operator has only discrete eigenvalues depending on the fiber parameter (*quasi-momentum*) called the branches. To prove (3.17) and (3.19) we first fix some notations.

Consider a 2-dimensional particle subjected to a constant magnetic field $\mathbf{B}=(0,0,\omega)$, which is orthogonal to the plane $\mathbb{R}^2$, where the particle is allowed to move. We use here the transverse (symmetric) gauge i.e. $\mathbf{a}(\mathbf{x})=\frac{1}{2}\mathbf{B}\wedge\mathbf{x}=\omega\mathbf{a}_0(\mathbf{x})=\omega/2(-x_2,x_1)$. Therefore, it is a two dimensional restriction of the model we consider in Section 3.2.

Take $\mathbf{e}_1$ and $\mathbf{e}_2$ as elements of the standard orthonormal basis in $\mathbb{R}^2$ and consider the lattice $\mathbb{Z}^2$. Let $\Xi=(-1/2,1/2)$, and denote the elementary cell in $\mathbb{R}^2$ by $\Xi\times\Xi$.

We denote the dual lattice by $(\mathbb{Z}^2)^*$, and define the dual elementary cell by

$$(\Xi^2)^*:=\{\mathbf{k}(\xi):=\xi_1\mathbf{k}_1+\xi_2\mathbf{k}_2,\ \xi=(\xi_1,\xi_2)\in\Xi^2\}=2\pi\,\Xi^2,$$

where $\mathbf{k}_{1,2}=2\pi\mathbf{e}_{1,2}$.

We suppose here that the magnetic field satisfies the following condition: there exists $N \in \mathbb{N}^*$ such that the magnetic flux through Ξ^2 is:

$$\mathbf{B} \cdot (\mathbf{e}_1 \wedge \mathbf{e}_2) = \omega |\mathbf{e}_1 \wedge \mathbf{e}_2| = 2\pi N \,. \tag{5.32}$$

It is well-known (see e.g. ref. 10) that for the particle restricted to the plane $\mathbb{R}^2$, the "free" magnetic Hamiltonian $h_0 = (-i\nabla - \mathbf{a})^2$ has only pure point spectrum (*Landau levels*), which is given by the set $\sigma(h_0) = \{(2n + 1)\omega \, : n \in \{0, 1, \dots\}\}$. Let $V \in C^0(\mathbb{R}^2)$ be a $\mathbb{Z}^2$-periodic external potential. For $\epsilon \geqslant 0$ the perturbed Hamiltonian

$$h_\epsilon = h_0 + \epsilon V$$

acts on $L^2(\mathbb{R}^2)$.

We now are interested in two questions: *first*, to justify the representation (3.17) and *second*, to investigate the nature of the spectrum of h_ϵ (in particular the bottom of it) and to elucidate (3.19).

5.2.1. Proof of 3.17

For $\mathbf{x}$, $\mathbf{y} \in \mathbb{R}^2$, define the "magnetic phase"

$$\phi(\mathbf{x}, \mathbf{y}) := -\frac{1}{2} \mathbf{B} \cdot (\mathbf{x} \wedge \mathbf{y}) \tag{5.33}$$

and recall that its main property is:

$$\begin{aligned} &\exp[-i\phi(\mathbf{x}, \mathbf{y})](-i\nabla_{\mathbf{x}} - \mathbf{a}(\mathbf{x})) \exp[i\phi(\mathbf{x}, \mathbf{y})] \\ &\quad = -i\nabla_{\mathbf{x}} - \mathbf{a}(\mathbf{x} - \mathbf{y}) \,. \end{aligned} \tag{5.34}$$

For all $\mathbf{e} = m\mathbf{e}_1 + n\mathbf{e}_2 \in \mathbb{Z}^2$, define $b(\mathbf{e}) := \pi N m n$. Let $\mathbf{f} = p\mathbf{e}_1 + q\mathbf{e}_2 \in \mathbb{Z}^2$. Then by virtue of (5.32)

$$\phi(\mathbf{e}, \mathbf{f}) = -\frac{1}{2} \mathbf{B} \cdot (\mathbf{e} \wedge \mathbf{f}) = \pi N (pn - qm) \,. \tag{5.35}$$

These imply that $b(\mathbf{e}) + b(\mathbf{f}) - b(\mathbf{e} + \mathbf{f}) - \phi(\mathbf{e}, \mathbf{f}) \in 2\pi\mathbb{Z}$, and that the *modified* magnetic translations:

$$(T_{\mathbf{e}}\psi)(\mathbf{x}) := \exp[i\phi(\mathbf{x}, \mathbf{e}) + ib(\mathbf{e})]\psi(\mathbf{x} - \mathbf{e}), \quad \mathbf{e} \in \Gamma, \quad \psi \in L^2(\mathbb{R}^2) \tag{5.36}$$

form an *abelian* group, i.e. $T_{\mathbf{e}} T_{\mathbf{f}} = T_{\mathbf{e}+\mathbf{f}}$. (Notice that we need the phase factor $b(\cdot)$ in magnetic translations because we work in the *symmetric* and not in the *Landau* gauge.)

Since (for any field) the magnetic translations (5.36) commute with the perturbed Hamiltonian h_ϵ, $\epsilon \geqslant 0$, it means that in the "rational case" a Bloch-Floquet decomposition must exist,[8,32] see also ref. 19 or recent paper,[12] for a brief account.

Now we explicitly decompose the operator h_ϵ into a direct fiber integral. This will be done first for h_0 (in fact for its resolvent) and then for h_ϵ. Define a direct fiber integral of $L^2(\Xi^2)$-spaces, $\mathcal{H} := \int_{\Xi^2}^{\oplus} L^2(\Xi^2)\, d\xi$, together with the unitary operator $U : L^2(\mathbb{R}^2) \mapsto \mathcal{H}$ whose action on smooth and compactly supported functions is:

$$(U\psi)(\xi, \underline{x}) = \sum_{\mathbf{e}\in\mathbb{Z}^2} \exp[-i\mathbf{k}(\xi)\cdot\mathbf{e} - i\phi(\underline{x}, \mathbf{e}) - ib(\mathbf{e})]\psi(\underline{x}+\mathbf{e}), \tag{5.37}$$

here $\underline{x}$ denotes the position variable $(x_1, x_2) \in \Xi^2$. Formula (5.37) is then extended by continuity on $L^2(\mathbb{R}^2)$. Its adjoint reads as ($\mathbf{e}\in\mathbb{Z}^2$):

$$\begin{aligned}&(U^*\psi)(\underline{x}+\mathbf{e})\\ &\quad = \int_{\Xi^2} d\xi' \exp[i\mathbf{k}(\xi')\cdot\mathbf{e} + i\phi(\underline{x}, \mathbf{e}) + ib(\mathbf{e})]\psi(\xi', \underline{x}).\end{aligned} \tag{5.38}$$

It is known (see refs. 9, 17 and references therein) that for z from the resolvent set $\rho(h_0)$, the resolvent $(h_0 - z)^{-1}$ admits the following integral kernel $K_0(\mathbf{x}, \mathbf{x}'; z)$:

$$\begin{aligned}K_0(\mathbf{x}, \mathbf{x}'; z) &= e^{i\phi(\mathbf{x},\mathbf{x}')}\, G_0(\mathbf{x}, \mathbf{x}'; z)\\ &\equiv \frac{\gamma(\alpha)}{4\pi} e^{i\phi(\mathbf{x},\mathbf{x}')}\, e^{-\psi(\mathbf{x},\mathbf{x}')}\mathcal{F}(\alpha, 1; 2\,\psi(\mathbf{x}, \mathbf{x}'))\end{aligned} \tag{5.39}$$

where $\psi(\mathbf{x}, \mathbf{x}') = \omega|\mathbf{x} - \mathbf{x}'|^2/4$, $\alpha = -(z/\omega - 1)/2 \neq -1, -2, \ldots$, γ is the Euler function, and $\mathcal{F}(\alpha, \beta; \zeta)$ is the confluent hyper-geometric function[1].

Take any $g \in C_0^\infty(\mathbb{R}^2)$. Since

$$\left(T_{\mathbf{e}}(h_0 - z)^{-1} g\right)(\mathbf{x}) = \left((h_0 - z)^{-1} T_{\mathbf{e}} g\right)(\mathbf{x}) \tag{5.40}$$

for any $\mathbf{x}\in\mathbb{R}^2$ and $\mathbf{e}\in\mathbb{Z}^2$, one has:

$$K_0(\mathbf{x}, \mathbf{x}' + \mathbf{e}; z)\exp(i\phi(\mathbf{x}', \mathbf{e})) = \exp(i\phi(\mathbf{x}, \mathbf{e}))K_0(\mathbf{x} - \mathbf{e}, \mathbf{x}'; z) \tag{5.41}$$

for any $\mathbf{x}'\in\mathbb{R}^2$ and for each $\mathbf{e}'\in\mathbb{Z}^2$ one has

$$K_0(\mathbf{x}, \underline{x}' + \mathbf{e}'; z)\exp(i\phi(\underline{x}', \mathbf{e}')) = \exp(i\phi(\mathbf{x}, \mathbf{e}'))K_0(\mathbf{x} - \mathbf{e}', \underline{x}'; z), \tag{5.42}$$

for any $\underline{x}' \in \Xi^2$ (notice that $\phi(\mathbf{x},\mathbf{x})=0$). Take a smooth $g \in \mathcal{H}$. Then by (5.38) and (5.42), we get

$$\{[(h_0-z)^{-1}]U^* g\}(\mathbf{x}) = \sum_{\mathbf{e}'\in\mathbb{Z}^2} \int_{\Xi^2} d\underline{x}' \int_{\Xi^2} d\xi' e^{i\mathbf{k}(\xi')\cdot\mathbf{e}'+i\phi(\mathbf{x},\mathbf{e}')+ib(\mathbf{e}')} \times K_0(\mathbf{x}-\mathbf{e}',\underline{x}';z) g(\xi',\underline{x}'). \tag{5.43}$$

Then with the help of (5.37) in the above equation, the expression for $\{U[(h_0-z)^{-1}]U^* g\}(\xi,\underline{x})$ reads as:

$$\sum_{\mathbf{e},\mathbf{e}'\in\mathbb{Z}^2} \int_{\Xi^2} d\underline{x}' \exp[-i\phi(\underline{x},\mathbf{e}) - ib(\mathbf{e}) + i\phi(\underline{x}+\mathbf{e},\mathbf{e}') + ib(\mathbf{e}')] \tag{5.44}$$
$$\times K_0(\underline{x}+\mathbf{e}-\mathbf{e}',\underline{x}';z)\exp[-i\mathbf{k}(\xi)\cdot\mathbf{e}] \int_{\Xi^2} d\xi' \exp[i\mathbf{k}(\xi')\cdot\mathbf{e}'] g(\xi',\underline{x}').$$

Changing the summation over the variable $\mathbf{e}$ to $\mathbf{f}=\mathbf{e}-\mathbf{e}'$, one gets for (5.43) that:

$$\sum_{\mathbf{e}'\in\mathbb{Z}^2}\sum_{\mathbf{f}\in\mathbb{Z}^2} \int_{\Xi^2} d\underline{x}' \exp[-i\phi(\underline{x},\mathbf{f}) - ib(\mathbf{e}'+\mathbf{f}) + i\phi(\mathbf{f},\mathbf{e}') + ib(\mathbf{e}')]$$
$$\times K_0(\underline{x}+\mathbf{f},\underline{x}';z)\exp[-i\mathbf{k}(\xi)\cdot\mathbf{f}]\exp[-i\mathbf{k}(\xi)\cdot\mathbf{e}']$$
$$\times \int_{\Xi^2} d\xi' \exp[i\mathbf{k}(\xi')\cdot\mathbf{e}'] g(\xi',\underline{x}'). \tag{5.45}$$

Since by the magnetic flux rationality (5.32) one has $-b(\mathbf{e}'+\mathbf{f})+\phi(\mathbf{f},\mathbf{e}')+b(\mathbf{e}')+b(\mathbf{f})\in 2\pi\mathbb{Z}$, and since

$$\sum_{\mathbf{e}'\in\mathbb{Z}^2} \exp[-i\mathbf{k}(\xi)\cdot\mathbf{e}'] \int_{\Xi^2} d\xi' \exp[i\mathbf{k}(\xi')\cdot\mathbf{e}'] g(\xi',\underline{x}') = g(\xi,\underline{x}'), \tag{5.46}$$

we conclude by (5.45) that the resolvent $(h_0-z)^{-1}$ is decomposable, and its fibers are integral operators on $L^2(\Xi^2)$ with kernels:

$$K_0(\xi;\underline{x},\underline{x}';z) = \sum_{\mathbf{f}\in\mathbb{Z}^2} \exp[-i\phi(\underline{x},\mathbf{f}) - ib(\mathbf{f}) - i\mathbf{k}(\xi)\cdot\mathbf{f}] K_0(\underline{x}+\mathbf{f},\underline{x}';z). \tag{5.47}$$

In order to obtain decomposition of the operator h_0 into fibers $h_0(\xi)$ which have a common domain (i.e. independent of ξ), one has to "rotate" $\mathcal{H}$ with the unitary operator $\mathcal{V}$ defined on each fiber by the multiplication $\mathcal{V}(\xi, \underline{x}) = \exp[-i\mathbf{k}(\xi)\cdot\underline{x}]$. Let $\mathcal{U} := \mathcal{V}U$. Then $\mathcal{U}(h_0 - z)^{-1}\mathcal{U}^* = \int_{\Xi^2} d\xi [h_0(\xi) - z]^{-1}$ and

$$[h_0(\xi) - z]^{-1}(\underline{x}, \underline{x}') = \exp[-i\mathbf{k}(\xi)\cdot\underline{x}] K_0(\xi; \underline{x}, \underline{x}'; z) \exp[i\mathbf{k}(\xi)\cdot\underline{x}']. \tag{5.48}$$

One can see that the range $[h_0(\xi) - z]^{-1} C_0^\infty(\Xi^2)$ is contained in the restriction to $\overline{\Xi^2}$ of all $C^\infty(\mathbb{R}^2)$-functions ψ with the property that both $\psi(\mathbf{x})$ and $[-i\nabla_{\mathbf{x}} - \mathbf{a}(\mathbf{x})]\psi(\mathbf{x})$ are invariant with respect to the modified magnetic translations (5.36). We say that the functions with such property verify "magnetic" *periodic boundary conditions*[8]. Denote this restriction by D. Then the operator $h_0(\xi) := [-i\nabla_{\underline{x}} - \mathbf{a}(\underline{x}) + \mathbf{k}(\xi)]^2$ with "magnetic" periodic boundary conditions is essentially self-adjoint on D. Now, if $\epsilon > 0$, everything remains true for h_ϵ, whose fibers are defined as operator sum:

$$h_\epsilon(\xi) = h_0(\xi) + \epsilon V(\underline{x}), \quad \underline{x} \in \Xi^2. \tag{5.49}$$

Then from (5.48) and (5.49) we derive (3.17).

5.2.2. Proof of 3.19

As it is well-known (see e.g. ref. 4), the orthogonal projectors of h_0 corresponding to the n-th Landau eigenvalue $\omega(2n+1)$ are integral operators, with the kernel (see also (5.39) for notations):

$$P_{0,n}(\mathbf{x}, \mathbf{x}') = \frac{\omega}{2\pi} e^{i\phi(\mathbf{x},\mathbf{x}')}\, e^{-\psi(\mathbf{x},\mathbf{x}')} \mathcal{L}_n(2\psi(\mathbf{x}, \mathbf{x}')), \tag{5.50}$$

where $\mathcal{L}_m(\zeta)$ is the m-th Laguerre polynomial, with $\mathcal{L}_m(0) = 1$, for any $m \geqslant 0$.

Then for each fiber $h_0(\xi)$ we have $h_0(\xi) = \sum_{n\geqslant 0} \omega(2n+1) P_{0,n}(\xi)$, where similar to (5.47) the "free" fiber projectors have the kernels:

$$P_{0,n}(\xi; \underline{x}, \underline{x}') = \sum_{\mathbf{f}\in\mathbb{Z}^2} e^{[-i\phi(\underline{x},\mathbf{f}) - ib(\mathbf{f}) - i\mathbf{k}(\xi)\cdot(\underline{x}+\mathbf{f}-\underline{x}')]} \times P_{0,n}(\underline{x} + \mathbf{f}, \underline{x}'; z). \tag{5.51}$$

Notice that the rank r of $P_{0,n}(\xi)$ can be easily obtained from:

$$\begin{aligned} r &= \int_{\Xi^2} d\underline{x}\, P_{0,n}(\xi; \underline{x}, \underline{x}) \\ &= \frac{\omega}{2\pi} \sum_{\mathbf{f}\in\mathbb{Z}^2} \int_{\Xi^2} d\underline{x}\, \exp[-2i\phi(\underline{x}, \mathbf{f}) - ib(\mathbf{f}) - i\mathbf{k}(\xi)\cdot\mathbf{f}] \\ &\quad \times \exp[-\omega\mathbf{f}^2/4]\, \mathcal{L}_n(\omega\mathbf{f}^2/2). \end{aligned} \tag{5.52}$$

Since $\mathbf{f} = f_1\mathbf{e}_1 + f_2\mathbf{e}_2 \in \mathbb{Z}^2$ and $\underline{x} = x_1\mathbf{e}_1 + x_2\mathbf{e}_2 \in \Xi^2$, then $-2i\phi(\underline{x}, \mathbf{f}) = 2i\pi N(f_1x_2 - f_2x_1)$. Therefore, the integral

$$\int_{\Xi^2} d\underline{x}\, \exp[-2i\phi(\underline{x}, \mathbf{f})] \tag{5.53}$$

is zero except for $\mathbf{f} = 0$, when it is equal to $|\Xi^2| = |\mathbf{e}_1 \wedge \mathbf{e}_2|$. Then by virtue of (5.32) we get $r = N$, for any $n \geqslant 0$. Then if $\epsilon > 0$ and small enough, by analytic perturbation theory one obtains that $h_\epsilon(\xi)$ has in the neighborhood of each Landau level $\omega(2n+1)$ exactly N discrete eigenvalues $\{\lambda_j^{(n)}(\epsilon, \xi)\}_{j=1}^N$. If $P_{\epsilon,n}(\xi)$ is the projector corresponding to each group of eigenvalues $\{\lambda_j^{(n)}(\epsilon, \xi)\}_{j=1}^N$ and if $\mathcal{S}_{\epsilon,n}(\xi)$ is the intertwining unitary:

$$\mathcal{S}_{\epsilon,n}(\xi) P_{\epsilon,n}(\xi) = P_{0,n}(\xi) \mathcal{S}_{\epsilon,n}(\xi)\,,$$

then after rotation by $\mathcal{S}_{\epsilon,n}(\xi)$ of the "reduced" perturbed Hamiltonian given by $P_{\epsilon,n}(\xi) h_\epsilon(\xi) P_{\epsilon,n}(\xi)$, one obtains that its eigenvalues

$$\{\lambda_j^{(n)}(\epsilon, \xi) =: \omega(2n+1) + \epsilon a_{n,j}(\epsilon, \xi)\}_{j=1}^N \tag{5.54}$$

are localized (up to an error of the order ϵ^2 and uniformly in ξ) in the neighborhood of the eigenvalues of the operator

$$L_{n,\epsilon}(\xi) := \omega(2n+1) P_{0,n}(\xi) + \epsilon P_{0,n}(\xi) V P_{0,n}(\xi)\,.$$

In the particular case of Section 3.2, when $N = 1$ (or $\omega = 2\pi$), one obtains that the operator $L_{n,\epsilon}(\xi)$ has only one eigenvalue which differs from $2\pi(2n+1)$ by $\epsilon a_{n,N=1}(\epsilon = 0, \xi)$, where (see (3.19))

$$
\begin{aligned}
a_n(\epsilon=0,\xi) &:= a_{n,N=1}(\epsilon=0,\xi) \\
&= \int_{\Xi^2} d\underline{x}\, V(\underline{x}) P_{0,n}(\xi;\underline{x},\underline{x}) \\
&= \sum_{\mathbf{f}\in\mathbb{Z}^2} \exp[-ib(\mathbf{f})-i\mathbf{k}(\xi)\cdot\mathbf{f}]\exp[-\pi\mathbf{f}^2/2]\mathcal{L}_n(\pi\mathbf{f}^2) \\
&\quad \times \int_{\Xi^2} d\underline{x}\, V(\underline{x})\exp[-2i\phi(\underline{x},\mathbf{f})].
\end{aligned} \tag{5.55}
$$

Denote by

$$
\hat{v}_{f_1,f_2} := \int_{\Xi^2} dx_1\, dx_2\, e^{-2\pi i(f_1x_1+f_2x_2)} V(x_1,x_2)
$$

the Fourier components of V $(f_1, f_2\in\mathbb{Z})$. Since $\omega=2\pi N$ and $N=1$ then

$$
\int_{\Xi^2} d\underline{x}\, V(\underline{x}) \exp[-2i\phi(\underline{x},\mathbf{f})] = \hat{v}_{f_2,-f_1}\,.
$$

By virtue of (5.55) we get that

$$
\begin{aligned}
&\int_{\Xi^2} d\xi\, |\nabla_\xi a_n(\epsilon=0,\xi)|^2 \\
&\quad = \sum_{\mathbf{f}\in\mathbb{Z}^2} \mathbf{f}^2\exp[-\pi\mathbf{f}^2/2]\mathcal{L}_n^2(\pi\mathbf{f}^2)|\hat{v}_{f_2,-f_1}|^2\,.
\end{aligned} \tag{5.56}
$$

If the above quantity is nonzero, then by analyticity of $a_n(\epsilon=0,\xi)$ one obtains that this function is not a constant. For example, if $n=0$, then $\mathcal{L}_1(\zeta)=1$ and (5.56) imply that any *nonconstant* potential ϵV transforms (at least for *small* ϵ) the unperturbed $(\epsilon=0)$ Landau fundamental state into a simple, *absolutely continuous* spectral band.

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